Functional symmetries and solutions of the dKP hierarchy

2004 J. Phys. A: Math. Gen. 37 5401

(http://iopscience.iop.org/0305-4470/37/20/011)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 5401–5416 PII: S0305-4470(04)73679-3

Functional symmetries and solutions of the dKP hierarchy

Francisco Guil

Departamento de Física Teórica, Facultad de Ciencias Físicas, Universidad Complutense, 28040 Madrid, Spain

Received 17 December 2003, in final form 23 March 2004
Published 5 May 2004
Online at stacks.iop.org/JPhysA/37/5401
DOI: 10.1088/0305-4470/37/20/011

Abstract

It is shown that the dispersionless KP hierarchy possesses an infinite collection of symmetries depending on arbitrary functions of the time variables. The twistor formulation of a factorization problem determines the family of symmetries and serves for the characterization of solutions of the hierarchy.

PACS numbers: 02.40.Ky, 02.30.Ik
Mathematics Subject Classification: 58B20

1. Introduction

The dispersionless limit of the Kadomtsev–Petviashvili hierarchy, the dKP hierarchy, admits a formulation as a system of Lax equations in a Lie algebra of functions, defined on a two-dimensional canonical manifold with commutator given by the Poisson bracket. For the origin and relevance of this dKP hierarchy see [1–14]. As for most systems of Lax equations, we can derive them from a factorization problem in the group corresponding to the given Lie algebra [12]. In the present situation such a formulation proves particularly useful when computing solutions of the dKP equation; the reason for this, in this case, is that the adjoint representation of the Lie group in the Lie algebra furnishes a tractable version of the factorization problem that characterizes the solutions to the dKP hierarchy. These are the twistor equations [11], the solutions of which can be conveniently described by means of the generating function formalism for the study of canonical transformations [11, 12].

The construction of solutions for the dKP hierarchy, as well as for other dispersionless equations, has been pursued along a number of different lines. To cite some of them, we recall the use of reductions of hydrodynamic type [13, 14], the hodograph solutions related to reductions given by systems of first-order partial differential equations [19, 20] and the dispersionless formulations of the inverse scattering [15, 16] and $\bar{\partial}$ method [17, 18]. We should also mention the connection of the dKP equation with the description of certain Einstein–Weyl spaces [21, 22], as well as the twistor equations associated with factorization problems in a group of canonical transformations we alluded to before.

In [12] we were able to construct solutions of the hierarchy containing arbitrary functions of the time variables. Such a construction was performed in a class of canonical transformations, defined in terms of certain generating functions as we said before. However, the functional dependence on the time variables, as observed in some concrete examples, can be compensated by an appropriate symmetry transformation depending on the given arbitrary functions. Such symmetries of the dKP equation, containing the first two times, were found in [21] associated with the Einstein–Weyl metric and characterized later in [12] as canonical transformations. It is the purpose of the present work to clarify the presence of such transformations in the whole dKP hierarchy, mainly because they represent a natural gauge when writing its solutions.

In this paper we shall obtain explicit formulae for the extension to the complete dKP hierarchy of these symmetry transformations. The construction of solutions given in [12] can thus be reduced to the study of simpler canonical transformations once we neglect the part of the canonical transformation responsible for the symmetry. We also give a simple test for the existence of solutions to the twistor equations and integral formulae for them.

The content and organization of this paper are as follows. In section 2 we examine the relevant facts about the dKP hierarchy that will be used in the rest of the work. We characterize, proposition 2.1, a class of twistor equations whose solutions admit a description by means of a finite number of conditions. Section 3 contains the construction of the solution, to the factorization problem in the group, for a family of canonical transformations with functional dependence in the time variables. These solutions reflect the ambiguity in writing solutions of the dKP hierarchy in their functional dependence on the time variables. This is the result used, in section 4, for the construction of the functional symmetries of the dKP hierarchy with the general formula for the transformation law of the solution to the dKP equation. Finally, in section 5 we describe the canonical transformations of [12] as a composition of simpler transformations and give formulae for the corresponding solutions.

2. The dKP hierarchy

The holomorphic cotangent bundle $E = T^*M \to M$ for the Riemann sphere $M = S^2$ is a two-dimensional complex manifold. The standard covering of $M = U_0 \bigcup U_1$ by open neighbours of zero and infinity gives coordinates (p, x) for points of E in terms of the differential $\mathrm{d}p$ of the coordinate p of M fixing the differential form $x\mathrm{d}p$ in T_p^*M. Sections of the exterior bundle $\bigwedge(E)$ are represented by the space of holomorphic differential forms $\Omega(E) = \Omega^0(E) \oplus \Omega^1(E) \oplus \Omega^2(E)$ where $\Omega^0(E) = \mathcal{O}_E$ is identified with the holomorphic sheaf of the manifold E.

The chosen coordinates for E define a volume form in the determinant bundle $\bigwedge^2(E)$ given by minus the differential of the canonical 1-form, $\alpha(E) = -\mathrm{d}(x\,\mathrm{d}p) = \mathrm{d}p \wedge \mathrm{d}x$ for $(p, x) \in U_0 \times \mathbf{C}$. Let f, g be two functions in $\mathcal{O}_E$; the relation

$$\mathrm{d}f \wedge \mathrm{d}g = \{f, g\}\alpha(E)$$

defines the commutator $\{f, g\}$. In coordinates

$$\{f, g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial p}.$$

Let D_0 and D_1 be two open discs in M centred at zero and infinity respectively, with non-empty intersection U for which we define the Lie algebra

$$g = \mathcal{O}_E(U \times \mathbf{C})$$

with the previous commutator. The Lie algebra g is the sum of two subalgebras $g = g_+ \oplus g_-$, where $g_+ = \mathcal{O}_E(D_0 \times \mathbf{C})$ and $g_- = \mathcal{O}_E(D_1 \times \mathbf{C})$ for functions vanishing at $p = \infty$. This is the situation described by the space of convergent Laurent series, in an annulus of the complex plane of the variable p, with coefficients depending on the variable x for the Lie algebra g. The subalgebras g_+ and g_- represent the positive and strictly negative parts of the series in each case.

Let L be an element in g that depends on new variables $t_2, t_3, \ldots$ and has the prescribed form [11, 12],

$$L(p, x, t_2, t_3, \ldots) = p + \sum_{j\geqslant 1} u_j(x, t_2, t_3, \ldots)p^{-j}.$$

As usual, define the functions P_n, $n = 2, 3, \ldots$, as the projections on g_+ of the positive powers of L, $P_n = L^n\,|_+$. The Lax–Sato hierarchy is given by the infinite set of equations

$$\frac{\partial L}{\partial t_n} = \{P_n, L\}$$

for $n = 2, 3, \ldots$, which in particular imply the dKP equation for the function $u = 2u_1$,

$$\left(u_t - \tfrac{3}{2}uu_x\right)_x = \tfrac{3}{4}u_{yy}$$

where u_1 is the coefficient of p^{-1} in L and we have set $t_2 = y, t_3 = t$. As functions of the time variables, $\mathbf{t} = (t_2, t_3, \ldots)$, we collect together the previous equations for L to write

$$\mathrm{d}L = \{\omega_+, L\}$$

in terms of the g_+-valued differential 1-form $\omega_+ = \sum_{n\geqslant 2} P_n\,\mathrm{d}t_n$ with $\mathrm{d}L = \sum \mathrm{d}t_n \partial L/\partial t_n$. The notation ω_+ suggests we define $\omega = \sum_{n\geqslant 2} L^n\,\mathrm{d}t_n$, the differential of which is given by $\mathrm{d}\omega = \sum_{n\geqslant 2}\{\omega_+, L^n\} \wedge \mathrm{d}t_n$, or equivalently

$$\mathrm{d}\omega = \{\omega_+, \omega\}.$$

Letting $\omega = \omega_+ - \omega_-$ represent the decomposition of the g-valued differential form ω into its positive and negative projections and taking into account the fact that $\{\omega, \omega\} = 0$, because $\{L^n, L^m\} = 0$, we obtain the zero-curvature equations

$$\mathrm{d}\omega_\pm = \tfrac{1}{2}\{\omega_\pm, \omega_\pm\} \tag{2.1}$$

as an equivalent description of the dKP hierarchy. This form of writing the system allows one for an integration procedure in the following terms. For a given element $H(p, x)$ in g, consider the associated Hamiltonian flow defined by the vector field $(-\partial H/\partial x, \partial H/\partial p)$,

$$\frac{\mathrm{d}p}{\mathrm{d}s} = -\frac{\partial H}{\partial x} \qquad \frac{\mathrm{d}x}{\mathrm{d}s} = \frac{\partial H}{\partial p}$$

which assigns to the initial point (p, x) at $s = 0$, the point $(p(s), x(s))$ given by the Taylor series in s

$$p(s) = \mathrm{Ad}\,\mathrm{e}^{sH}p \qquad x(s) = \mathrm{Ad}\,\mathrm{e}^{sH}x.$$

Here we define the adjoint action according to

$$\mathrm{Ad}\,\mathrm{e}^{sH} f(p, x) = \mathrm{e}^{\mathrm{ad}\,sH} f(p, x)$$

given in terms of the adjoint operator in the Lie algebra $(\mathrm{ad} f)g = \{f, g\}$. The exponential map, for the group of canonical transformations associated with the Lie algebra g, assigns to H the group element $\exp H$ that corresponds to the value of the flow defined by H at $s = 1$ for a smooth Hamiltonian H. The relevance of that group in the dKP hierarchy stems from the fact that the function $L(p, x)$ can be described by the action of the flow defined by a generic Hamiltonian $H_-(p, x) = \sum_{j\geqslant 1} H_j(x)p^{-j}$ in g_-; we have

$$L(p, x) = \mathrm{Ad}\,\mathrm{e}^{H_-(p,x)}p \tag{2.2}$$

where the dependence on the time variables of both L and H_- is not explicitly displayed. The integration of the Lax–Sato equations for the dKP hierarchy starts with the observation that for such $L(p, x)$, the differential in the time variables takes the form $\mathrm{d}L = \{\omega_-, L\}$. The 1-form $\omega_- = \mathrm{d}\psi_-\psi_-^{-1}$ is defined by the right differential of the group element $\psi_- = \exp H_-$,

$$\mathrm{d}\psi_-\psi_-^{-1} = \sum_{k\geqslant 0} \frac{(\mathrm{ad}H_-)^k}{(k+1)!}\,\mathrm{d}H_-.$$

The differential form ω becomes in that case

$$\omega = \sum_{n\geqslant 2} L^n\,\mathrm{d}t_n = \mathrm{Ad}\psi_-\,\mathrm{d}t(p)$$

with $\mathrm{d}t(p) = \sum_{n\geqslant 2} p^n\,\mathrm{d}t_n$. The decomposition into its positive and negative projections leads to the condition that the form

$$\omega_+ = \mathrm{d}\psi_-\psi_-^{-1} + \mathrm{Ad}\psi_-\,\mathrm{d}t(p)$$

must be positive and of zero curvature in order to the function $L(p, x)$ be a solution of the dKP hierarchy. Accordingly, we write the solution to the zero-curvature equation as $\omega_+ = \mathrm{d}\psi_+\psi_+^{-1}$. In that case, $\psi_+ = \exp H_+$ where H_+ takes values in g_+. Then we see that the relation in the Lie algebra may be formulated as an equation in the group,

$$\psi_-\,\mathrm{e}^{t(p)}k = \psi_+ \tag{2.3}$$

that follows from the fact that the two elements in the group from which one gets ω_+ can differ at most in a constant element k, independent of the time variables. Condition (2.3) admits a reformulation, as a factorization problem in the group for the flow $\mathrm{e}^{t(p)}k$ associated with an arbitrary transformation k, given by the equation

$$\mathrm{e}^{t(p)}k = \psi_-^{-1}\psi_+. \tag{2.4}$$

The relevance of this relation is that it allows us to describe the transformations ψ_- for which the function $L(p, x)$ in (2.2) is a solution for the dKP hierarchy.

Besides the function $L(p, x)$, the function

$$M(p, x) = \mathrm{Ad}\,\mathrm{e}^{H_-(p,x)}\mathrm{Ad}\,\mathrm{e}^{t(p)}x \tag{2.5}$$

proves to be a basic ingredient in the dKP theory [11]. First, observe that $M = \mathrm{Ad}\,\mathrm{e}^{H_-}(x + \partial t(p)/\partial p)$; the pair L, M is then a canonical pair obtained from $(p, x + t'(p))$ through the action of $\exp H_-$. That is to say that they are the solution of

$$\frac{\mathrm{d}p}{\mathrm{d}s} = -\frac{\partial H_-}{\partial x} \qquad \frac{\mathrm{d}x}{\mathrm{d}s} = \frac{\partial H_-}{\partial p}$$

issued from $(p(0) = p, x(0) = x + t'(p))$ at $s = 1$ $(L = p(1), M = x(1))$. Therefore, for the evolution defined by H_- we have the relation $H_-(p, x + t'(p)) = H_-(L, M)$ or

$$\mathrm{Ad}\,\mathrm{e}^{t(p)}H_-(p, x) = H_-(L, M)$$

for the first integral H_-, which shows another aspect of the connection between the pair (L, M) and the Hamiltonian H_-. An alternative description of the canonical variables (L, M) comes from the generating function formalism that defines the new pair by the condition

$$M\,\mathrm{d}L + p\,\mathrm{d}x = \mathrm{d}\Phi(L, x)$$

which amounts to the implicit system for (L, M)

$$p = \frac{\partial \Phi}{\partial x}(L, x) \qquad M = \frac{\partial \Phi}{\partial L}(L, x). \tag{2.6}$$

In these formulae the generating function $\Phi(L, x)$ is to be understood as a given function of the form

$$\Phi(L, x) = t(L) + xL + \phi(L, x)$$

as indicated by the previous expressions for L and M with

$$\phi(L, x) = \sum_{j\geqslant 1} \phi_j(x) L^{-j}.$$

When combined with the adjoint representation of the factorization problem (2.4) one obtains the twistor equations for the solutions of the dKP hierarchy in the following terms. Let $P(p, x) = \mathrm{Ad}\, kp$ and $X(p, x) = \mathrm{Ad}\, kx$ be the two canonical variables defined by the constant element k. The definitions (2.2) and (2.5) of L and M respectively lead to the equivalent form of the factorization problem (2.4):

$$P(L, M) = \mathrm{Ad}\psi_+ p \qquad X(L, M) = \mathrm{Ad}\psi_+ x.$$

From these equations we deduce that the negative parts of $P(L, M)$ and $X(L, M)$, when expressed in powers of the variable p, must vanish

$$P(L, M)|_- = 0 \qquad X(L, M)|_- = 0. \tag{2.7}$$

The generating function $\Phi(L, x)$ reduces the problem to a pair of equations for the partial derivatives of such function

$$P(L, \Phi_L) = F_+(\Phi_x, x) \qquad X(L, \Phi_L) = G_+(\Phi_x, x) \tag{2.8}$$

where $F_+(p, x)$ and $G_+(p, x)$ are the positive parts of $P(L, M)$ and $X(L, M)$ when both L and M are expressed in terms of p and x through (2.6). We define $F_-(p, x)$ and $G_-(p, x)$ with the same meaning for the negative parts. The existence of a solution, provided we can solve these equations for the partial derivatives of Φ, is guaranteed by the compatibility conditions

$$\{P, X\}\Phi_{L,x} = \{F_+, G_+\}\Phi_{x,L}$$

one gets by taking partial derivatives. But, by construction, P and X are a pair of canonical variables with commutator $\{P, X\} = 1$. For a solution Φ_L, Φ_x the same is true for $F_+ = P$ and $G_+ = X$ from which we have $\{F_+, G_+\} = 1$ and hence the mixed partial derivatives of Φ coincide. A solution $\Phi(L, x)$ serves as a generating function for a solution $L(p, x)$ of the dKP hierarchy provided it depends on L and x in the form we assumed previously. A useful application of this compatibility relation appears in connection with the reduction of system (2.7) to a system with a finite number of equations.

Proposition 2.1. *Assume the first of the equations in system (2.7) can be solved so that $P = F_+$ for some polynomial $F_+(p, x)$ of degree m in the variable p. Then, the vanishing of the first $m - 1$ coefficients in the expansion of G_- in powers of p implies the condition $\{F_+, G_+\} = 1$.*

Proof 2.1. Because $\{P, X\} = 1$ and $P = F_+$ we will have $\{F_+, G_+\} - 1 = \{F_+, G_-\}$ according to the decomposition of G into positive and negative parts. In this last relation the member on the left-hand side is positive, therefore so must be the bracket on the right-hand side whose positive part is equal to zero since the series for G_- begins with a term of the form $b(x)p^{-m}$ and the result follows. □

3. Special solutions

Among the canonical transformations studied in [12], which furnish examples of variables (P, X) for the twistor equations, we shall consider the particular case defined by a generating function of the form

$$J(P, \rho) = J_{r+1}(\rho) P^{r+1} \tag{3.1}$$

where for each fixed integer $r \geqslant 2$, $J_{r+1}(\rho)$ represents an arbitrary analytical function at $\rho = 0$. The value of the integer r also fixes the number of time variables entering in the function

$$t(p) = t_2 p^2 + \cdots + t_{r+1} p^{r+1}.$$

The variables (P, X) are defined through the intermediate canonical pair (h, ρ),

$$h = p^{r+1} \qquad \rho = \frac{x}{(r+1)p^r}$$

by means of the composition,

$$(p, x) \to (h, \rho) \to (P, X)$$

when we compute the differential of the generating function (3.1)

$$\mathrm{d}J = X\,\mathrm{d}P + h\,\mathrm{d}\rho.$$

In that case the formulae $X = \partial J/\partial P$, $h = \partial J/\partial \rho$ give the equations

$$p^{r+1} = J'_{r+1}(\rho) P^{r+1} \qquad X = (r+1) J_{r+1}(\rho) P^r$$

which admit a solution

$$P = \frac{1}{(J'_{r+1})^{1/(r+1)}} p \qquad X = \frac{(r+1) J_{r+1}}{(J'_{r+1})^{r/(r+1)}} p^r. \tag{3.2}$$

To go over the associated equations (2.8), we construct the functions

$$F(L, x) = \frac{1}{(J'_{r+1})^{1/(r+1)}} L \qquad G(L, x) = \frac{(r+1) J_{r+1}}{(J'_{r+1})^{r/(r+1)}} L^r$$

as well as a function $\rho(L, x)$ defined by

$$\rho(L, x) = \frac{1}{(r+1)L^r}(t'(L) + x + \phi_L).$$

The positive part for $F(L, x)$ is then

$$F_+(p, x) = \frac{p}{(J'_{r+1}(t_{r+1}))^{1/(r+1)}} - \frac{r t_r J''_{r+1}(t_{r+1})}{(r+1)^2 (J'_{r+1}(t_{r+1}))^{\frac{r+2}{r+1}}} \tag{3.3}$$

which we abbreviate as $F_+(p, x) = f_0(t_{r+1}) p + f_1(t_{r+1}, t_r)$. Analogously, we encounter for $G_+(p, x)$ an expression of the following type:

$$G_+(p, x) = g_0(t_{r+1}) P_r + g_1(t_{r+1}, t_r) P_{r-1} + \cdots + g_r(t_{r+1}, \ldots, t_2, x) \tag{3.4}$$

where, as we did previously, we denote $P_k = L^k|_+$. The first of the equations (2.8), $F(L, x) = F_+(L, x)$, is enough in this case to determine the solution. In fact we have

$$\frac{L}{(J'_{r+1}(\rho))^{1/(r+1)}} = f_0(t_{r+1}) p + f_1(t_{r+1}, t_r)$$

with $\rho = \rho(L, x)$ as before. Since $p = L + \phi_x$, equation (2.6), we come to the relation

$$\frac{1}{f_0}\left(\frac{L}{(J'_{r+1}(\rho))^{1/(r+1)}} - f_1\right) = L + \phi_x$$

which determines $\phi_{1,x}, \ldots, \phi_{r,x}$ from the series expansion at $L = \infty$. From $p = L + \phi_x$ and successive powers $p^2 = L^2 + 2L\phi_x + \phi_x^2, \ldots$ we obtain the polynomials $P_2 = L^2|_+, \ldots, P_{r+1} = L^{r+1}|_+$ needed for the construction of the solution to the dKP hierarchy, expressed by the vanishing of the curvature for the connection

$$\omega_+ = P_2\,\mathrm{d}t_2 + \cdots + P_{r+1}\,\mathrm{d}t_{r+1}. \tag{3.5}$$

In the simplest cases, $r = 2, 3$, the foregoing expressions yield, denoting $z = J_{r+1}(t_{r+1})$, $t_3 = t$, $t_2 = y$, the formula

$$\omega_+ = \left[p^2 + \frac{6xz'' + 4y^2z^{(3)}}{27z'} - \frac{16y^2(z'')^2}{81(z')^2}\right]\mathrm{d}y + \left[p^3 + p\left[\frac{3xz'' + 2y^2z^{(3)}}{9z'} - \frac{8y^2(z'')^2}{27(z')^2}\right]\right. \\ \left. + \frac{18xyz^{(3)} + 4y^3z^{(4)}}{81z'} - \frac{24xy(z'')^2 + 16(y)^3z''z^{(3)}}{81(z')^2} + \frac{112y^3(z'')^3}{729(z')^3}\right]\mathrm{d}t$$

for $r = 2$. If, instead, we take $r = 3$ we shall obtain the solution

$$\omega_+ = P_2\,\mathrm{d}t_2 + P_3\,\mathrm{d}t_3 + P_4\,\mathrm{d}t_4$$

with

$$P_2 = p^2 + \frac{yz''}{4z'} + \frac{9t^3[-5(z'')^2 + 4z'z^{(3)}]}{256(z')^2}.$$

We also deduce the formula

$$P_3 = p^3 + \frac{p}{8192(z')^3}[3072y(z')^2z'' - 2160t^2z'(z'')^2 + 1728t^2(z')^2z^{(3)}] \\ + \frac{1}{8192(z')^3}[1536x(z')^2z'' - 2880ytz'(z'')^2 + 1215t^3(z'')^3 \\ + 2304yt(z')^2z^{(3)} - 1620t^3z'z''z^{(3)} + 432t^3(z')^2z^{(4)}]$$

and finally we get the following expression for P_4:

$$P_4 = p^4 + \frac{1}{65536(z')^4}\{512p^2[64y(z')^3z'' - 45t^2(z')^2(z'')^2 + 36t^2(z')^3z^3] \\ + 32p[512x(z')^3z'' - 960yt(z')^2(z'')^2 + 405t^3z'(z'')^3 + 768yt(z')^3z^{(3)} \\ - 540t^3(z')^2z''z^{(3)} + 144t^3(z')^3z^{(4)}] - 1024[8y^2 + 15xt](z')^2(z'')^2 \\ + 4096[2y^2 + 3xt](z')^3z^{(3)} + 4608yt^2[5z'(z'')^3 - 7(z')^2z''z^{(3)} \\ + 2(z')^3z^{(4)}] + 27t^4[-255(z'')^4 + 480z'(z'')^2z^{(3)} - 96(z')^2(z^{(3)})^2 \\ - 160(z')^2z''z^{(4)} + 32(z')^3z^{(5)}]\}.$$

Let now ξ_+ be the group element defined by the product

$$\xi_+ = \mathrm{e}^{\gamma p x}\,\mathrm{e}^{\beta x}\,\mathrm{e}^{A} \tag{3.6}$$

where $\gamma = \gamma(\mathbf{t})$ and $\beta = \beta(\mathbf{t})$ depend on the first r time variables. Here it is understood that $\mathbf{t} = (t_2, \ldots, t_{r+1})$, while $A = A(p, \mathbf{t})$ is a positive function on p independent of the coordinate x. The transformation given by formulae (3.2), represented in the group by a constant element k_1, gives rise to a factorization problem, equation (2.4),

$$\mathrm{e}^{t(p)}k_1 = \xi_-^{-1}\xi_+$$

whose solution for the positive part ξ_+ is precisely an element of the form (3.6) upon an appropriate choice of the functions contained in it. This will be the case provided $F_+(p, x)$ and $G_+(p, x)$, in formulae (3.3) and (3.4) respectively, are related to ξ_+ in accordance with equations (2.8),

$$\mathrm{Ad}\xi_+ p = F_+(p, x) \qquad \mathrm{Ad}\xi_+ x = G_+(p, x). \tag{3.7}$$

The action of ξ_+ on the canonical coordinates reduces to the composition of three flows with Hamiltonians $H_1 = A(p, \mathbf{t})$, $H_2 = x$ and $H_3 = px$. The solution to Hamilton's equations defines, as in section 2, the adjoint action of the element corresponding to the Hamiltonian in each case. Thus, for $H_1 = A(p, \mathbf{t})$ the solution to Hamilton's equations $\dot{p} = 0$, $\dot{x} = A'(p, \mathbf{t})$, with $A'(p, \mathbf{t}) = \partial A(p, \mathbf{t})/\partial p$, is given by the formulae

$$p(s) = p \qquad x(s) = x + sA'(p, \mathbf{t})$$

which taken at $s = 1$ determine the adjoint action of e^A according to the formulae

$$\mathrm{Ad}\,\mathrm{e}^A p = p \qquad \mathrm{Ad}\,\mathrm{e}^A x = x + A'(p, \mathbf{t}).$$

For $H_2 = x$, the equations $\dot{p} = -1$, $\dot{x} = 0$ have the solution $p(s) = p - s$, $x(s) = x$, which at $s = \beta$ define

$$\mathrm{Ad}\,\mathrm{e}^{\beta x} p = p - \beta \qquad \mathrm{Ad}\,\mathrm{e}^{\beta x} x = x.$$

When we consider $H_3 = px$, from $\dot{p} = -p$, $\dot{x} = x$ we deduce the solution

$$p(s) = p\,\mathrm{e}^{-s} \qquad x(s) = x\,\mathrm{e}^{s}$$

which for $s = \gamma$ are

$$\mathrm{Ad}\,\mathrm{e}^{\gamma px} p = p\,\mathrm{e}^{-\gamma} \qquad \mathrm{Ad}\,\mathrm{e}^{\gamma px} x = x\,\mathrm{e}^{\gamma}.$$

Taken together these formulae we obtain the composition

$$\mathrm{Ad}\xi_+ p = \mathrm{Ad}\,\mathrm{e}^{\gamma px}\,\mathrm{Ad}\,\mathrm{e}^{\beta x}\,\mathrm{Ad}\,\mathrm{e}^{A}\, p$$

from which we get

$$\mathrm{Ad}\xi_+ p = \mathrm{Ad}\,\mathrm{e}^{\gamma px}(p - \beta) = p\,\mathrm{e}^{-\gamma} - \beta$$

as follows from our previous computations. Analogously, for the action on the coordinate x we have

$$\mathrm{Ad}\xi_+ x = \mathrm{Ad}\,\mathrm{e}^{\gamma px}\,\mathrm{Ad}\,\mathrm{e}^{\beta x}\,\mathrm{Ad}\,\mathrm{e}^{A}\, x$$

which gives

$$\mathrm{Ad}\xi_+ x = \mathrm{Ad}\,\mathrm{e}^{\gamma px}\,\mathrm{Ad}\,\mathrm{e}^{\beta x}(x + A'(p, \mathbf{t})) = \mathrm{Ad}\,\mathrm{e}^{\gamma px}(x + A'(p - \beta, \mathbf{t}))$$

from which we conclude that

$$\mathrm{Ad}\xi_+ x = x\,\mathrm{e}^{\gamma} + A'(p\,\mathrm{e}^{-\gamma} - \beta, \mathbf{t}).$$

The first of conditions (3.7), due to equation (3.3) and the formulae for the adjoint action of the element ξ_+, becomes

$$\mathrm{e}^{-\gamma} p - \beta = \frac{p}{(J'_{r+1}(t_{r+1}))^{1/(r+1)}} - \frac{r t_r J''_{r+1}(t_{r+1})}{(r+1)^2 (J'_{r+1}(t_{r+1}))^{\frac{r+2}{r+1}}}$$

from which we obtain the functions γ and β according to the following formulae:

$$\mathrm{e}^{-\gamma} = \frac{1}{(J'_{r+1}(t_{r+1}))^{1/(r+1)}} \qquad \beta = \frac{r t_r J''_{r+1}(t_{r+1})}{(r+1)^2 (J'_{r+1}(t_{r+1}))^{\frac{r+2}{r+1}}}.$$

In addition, from (3.7) and the action of ξ_+ on x we have

$$x\,\mathrm{e}^{\gamma} + A'(\mathrm{e}^{-\gamma} p - \beta, \mathbf{t}) = G_+(p, x).$$

Note that since $\{F_+, G_+\} = 1$, the expression found for F_+ implies that

$$G_+(p, x) - x\,\mathrm{e}^{\gamma}$$

does not depend on the coordinate x. The substitution $\mathrm{e}^{-\gamma}p-\beta=q$ gives the equation for $A(q,\mathbf{t})$

$$\frac{\partial A}{\partial q}(q,\mathbf{t})=G_+(\mathrm{e}^{\gamma}(\beta+q),x)-x\,\mathrm{e}^{\gamma}$$

the solution of which is

$$A(q,\mathbf{t})=A_0(\mathbf{t})+\int \mathrm{d}q(G_+(\mathrm{e}^{\gamma}(\beta+q),x)-x\,\mathrm{e}^{\gamma}).$$

The integration constant $A_0(\mathbf{t})$, which is a function of the time variables, cannot be determined from (3.7) solely; we need the right differential of ξ_+ with respect to the time variables which is precisely the zero-curvature form ω_+ in (3.5) we have already explicitly computed in some examples. For such differential we get

$$\mathrm{d}\xi_+\xi_+^{-1}=px\,\mathrm{d}\gamma+x\mathrm{e}^{\gamma}\,\mathrm{d}\beta+\mathrm{d}A(\mathrm{e}^{-\gamma}p-\beta,\mathbf{t}) \tag{3.8}$$

and $A_0(\mathbf{t})$ follows from the condition $\mathrm{d}\xi_+\xi_+^{-1}=\omega_+$ at $p=0$. The case $r=2$, for instance, gives the expressions

$$\gamma=\frac{1}{3}\log z' \qquad \beta=\frac{2yz''}{9(z')^{4/3}}$$

where we continue with the abbreviated notation $t_2=y, t_3=t, z=J_3(t)$. The element ξ_+ of (3.6) is finally determined by

$$A(p,y,t)=zp^3+\frac{y[9p(z')^{4/3}+2yz'']p}{9(z')^{2/3}}+\frac{4y^3[z'z^{(3)}-(z'')^2]}{81(z')^2}.$$

Analogously, for $r=3$ and letting z now denote the arbitrary function $z=J_4(t_4)$, we arrive at the expressions

$$\gamma=\frac{1}{4}\log z' \qquad \beta=\frac{3tz''}{16(z')^{5/4}}$$

besides the polynomial

$$\begin{aligned}A(p,y,t,t_4)&=zp^4+t(z')^{3/4}p^3+\frac{(32yz'+9t^2z'')p^2}{32(z')^{1/2}}\\&\quad+\frac{3p}{512(z')^{7/4}}[64ytz'z''+3t^3(-3(z'')^2+4z'z^{(3)})]\\&\quad+\frac{1}{2048(z')^3}[256y^2(z')^2z''+288yt^2z'(z'z^{(3)}-(z'')^2)\\&\quad+27t^4(2(z'')^3-3z'z''z^{(3)}+(z')^2z^{(4)})].\end{aligned}$$

Most of our previous discussion remains unchanged when we replace the generating function (3.1), for the canonical transformation we have been dealing with, by the more general function

$$J(P,\rho)=\sum_{k=1}^{r+1}J_k(\rho)P^k \tag{3.9}$$

already studied in [12]. With the same notation conventions we have used up to now, we shall have canonical variables (P,X) defined by the implicit relations

$$p^{r+1}=\sum_{k=1}^{r+1}J_k'(\rho)P^k \qquad X=\sum_{k=1}^{r+1}kJ_k(\rho)P^{k-1}$$

which admit a solution given by power series in p. Functions $F(L, x)$ and $G(L, x)$ are now determined by the equations

$$L^{r+1} = \sum_{k=1}^{r+1} J_k'(\rho(L, x))F(L, x)^k \qquad G(L, x) = \sum_{k=1}^{r+1} kJ_k(\rho(L, x))F(L, x)^{k-1}$$

with the same function $\rho(L, x)$ as before. From what we have said, it readily follows that the positive parts are again polynomials

$$F_+(p, x) = f_0 + f_1 p \qquad G_+(p, x) = \sum_{k=0}^{r} g_k p^k \tag{3.10}$$

which imply the twistor equations (2.8) for ϕ_x and ϕ_L,

$$L^{r+1} = \sum_{k=1}^{r+1} J_k'(\rho(L, x))[f_0 + (L + \phi_x) f_1]^k \tag{3.11}$$

$$\sum_{k=0}^{r} g_k (L + \phi_x)^k = \sum_{k=1}^{r+1} kJ_k(\rho(L, x))[f_0 + (L + \phi_x) f_1]^{k-1}. \tag{3.12}$$

We do not try to write the lengthy explicit expressions for the zero-curvature form ω_+ induced by the function (3.9) as we did for the particular and simpler case (3.1). In the present situation (3.9) and for $r = 2$, all the information needed is encoded in the first two coefficients of the series expansion for p in negative powers of L according to (2.6), $p = L + \phi_{1,x} L^{-1} + \phi_{2,x} L^{-2} + \cdots$; note that the function $u(x, y, t) = -2\phi_{1,x}$ solves the dKP equation. We set $z_a = J_a(t)$ for $a = 1, 2, 3$ in terms of which we obtain the formulae

$$\phi_{1,x} = \frac{1}{9(z_3')^{4/3}}[(z_2')^2 - 3z_1' z_3' - x(z_3')^{1/3} z_3''] - \frac{2y[z_3' z_2'' - z_2' z_3'']}{9(z_3')^{5/3}} + \frac{y^2[8(z_3'')^2 - 6z_3'(z_3)^{(3)}]}{81(z_3')^2} \tag{3.13}$$

and

$$\begin{aligned}\phi_{2,x} = {} & \frac{1}{81(z_3')^2}[-2(z_2')^3 - 9x(z_3')^{4/3} z_2'' + 9z_2'[z_1' z_3' + x(z_3')^{1/3} z_3'']] \\ & - \frac{2y}{81(z_3')^{7/3}}[9(z_3')^2 z_1'' + 5(z_2')^2 z_3'' - 4x(z_3')^{1/3}(z_3'')^2 \\ & - 6z_3'(z_2' z_2'' + z_1' z_3'') + 3x(z_3')^{4/3}(z_3)^{(3)}] + \frac{y^2}{27(z_3')^{8/3}}[-4z_2'(z_3'')^2 \\ & - 2(z_3')^2 (z_2)^{(3)} + 2z_3'[2z_2'' z_3'' + z_2'(z_3)^{(3)}]] \\ & - \frac{4y^3}{2187(z_3')^3}[28(z_3'')^3 - 36z_3' z_3''(z_3)^{(3)} + 9(z_3')^2 (z_3)^{(4)}].\end{aligned}$$

The positive part ξ_+ of the solution to the factorization problem in the present case is obtained from conditions (3.7) for the solutions $F_+(p, x)$ and $G_+(p, x)$ to equations (3.11) and (3.12).

Proposition 3.1. *The group element ξ_+ corresponding with the solution of the factorization problem (2.4) for the canonical transformation associated with the generating function (3.9) admits a representation in the form*

$$\xi_+ = e^{\gamma p x}\, e^{\beta x}\, e^{A}$$

where γ and β depend on the time variables $\mathbf{t} = (t_2, \dots, t_{r+1})$ while A is a polynomial of degree $r+1$ in p whose coefficients are functions of $\mathbf{t}$.

Note that the solution of the dKP hierarchy $\omega_+ = \mathrm{d}\xi_+\xi_+^{-1}$ is given by a g-valued zero-curvature form that as in (3.8) is defined in a finite-dimensional subalgebra of the Lie algebra g_+. Namely, the Lie algebra of polynomial functions in (p, x) of the form

$$h(p,x) = h_0(p) + xh_1(p)$$

with coefficients given by polynomials in p, $h_0(p)$ of degree $r+1$ and $h_1(p)$ of degree 1.

4. Symmetries

As we have seen, the generating function (3.9) gives rise to a solution of the dKP hierarchy that follows from the factorization problem

$$\mathrm{e}^{t(p)}k_1 = \xi_-^{-1}\xi_+ \tag{4.1}$$

where k_1 represents the flow that induces the transformation defined by the function (3.9). Let k denote a group element for which, by composition with (4.1), we obtain

$$\mathrm{e}^{t(p)}k_1 k = \xi_-^{-1}\xi_+ k$$

besides the associated factorization problem for $k_1 k$

$$\mathrm{e}^{t(p)}k_1 k = \psi_-^{-1}\psi_+. \tag{4.2}$$

This gives, in fact, a description of the factorization

$$\xi_+ k = \eta_-^{-1}\eta_+ \tag{4.3}$$

as $\psi_- = \eta_-\xi_-$ and $\psi_+ = \eta_+$. Thus we see that the zero-curvature form

$$\omega_+ = \mathrm{d}\eta_+\eta_+^{-1} = \mathrm{d}\psi_+\psi_+^{-1} \tag{4.4}$$

represents a solution of the dKP hierarchy that corresponds precisely to the factorization problem for the element $k_1 k$ and gives a solution for the dKP equation $u(x, \mathbf{t})$ in coordinates x and $\mathbf{t} = (t_2, \dots, t_{r+1})$. We shall describe the solution ω_+ in (4.4) in terms of the solution to the dKP hierarchy $\mathrm{d}\xi_+\xi_+^{-1}$, obtained from (4.1), we have considered in the previous section. Since $\eta_+ = \eta_-\xi_+ k$, as follows from (4.3), the positive part of the right differential results in an expression for ω_+ as

$$\omega_+ = \mathrm{Ad}\eta_-\, \mathrm{d}\xi_+\xi_+^{-1}\big|_+ = \mathrm{d}\xi_+\xi_+^{-1} + \left\{\ln\eta_-, d\xi_+\xi_+^{-1}\right\}_+ + \cdots \tag{4.5}$$

which differs from the usual solutions in the replacement of $\exp t(p)$ for ξ_+ in the factorization problem. Due to the special structure of the element ξ_+ figuring in proposition (3.1) we shall see that there exist new coordinates $\tilde{x}, \tilde{\mathbf{t}}$ in terms of which the factorization problem (4.3) becomes an ordinary factorization problem for $\exp \tilde{t}(\tilde{p})$.

To begin with, let us define new canonical coordinates $\tilde{p}, \tilde{x}$ and new time variables $\tilde{\mathbf{t}}$ by means of the Lie algebra automorphism (canonical transformation) induced by the adjoint action of the given element ξ_+,

$$\mathrm{Ad}\xi_+ p = \mathrm{Ad}\,\mathrm{e}^{\tilde{t}(\tilde{p})}\tilde{p} = \tilde{p} \tag{4.6}$$

$$\mathrm{Ad}\xi_+ x = \mathrm{Ad}\,\mathrm{e}^{\tilde{t}(\tilde{p})}\tilde{x} = \tilde{x} + \frac{\partial\tilde{t}}{\partial\tilde{p}}. \tag{4.7}$$

We have thus obtained the formulae

$$\tilde{p} = F_+(p,x) \qquad \tilde{x} + \frac{\partial\tilde{t}}{\partial\tilde{p}} = G_+(p,x) \tag{4.8}$$

by replacement of (3.10) in (4.6) and (4.7) respectively. The factorization problem (4.3) generates the equations in the Lie algebra

$$\mathrm{Ad}\xi_+ P(p,x) = \mathrm{Ad}(\eta_-^{-1}\eta_+)p$$

and

$$\mathrm{Ad}\xi_+ X(p,x) = \mathrm{Ad}(\eta_-^{-1}\eta_+)x$$

if (P, X) is obtained from (p, x) acting with k. Since, as follows from (4.6) and (4.7)

$$\mathrm{Ad}\xi_+ P(p,x) = \mathrm{Ad}\,\mathrm{e}^{\tilde{t}(\tilde{p})} P(\tilde{p},\tilde{x})$$

and the corresponding formula for $X(p, x)$, we see that equation (4.3), when written in the new tilde coordinates, reads

$$\mathrm{e}^{\tilde{t}(\tilde{p})}\tilde{k} = \tilde{\eta}_-^{-1}\tilde{\eta}_+.$$

In this relation, the tilde transformations $\tilde{\eta}_-, \tilde{\eta}_+$ are defined by substitution of p, x in terms of $\tilde{p}, \tilde{x}$, according to (4.8), in the elements η_-, η_+. Note that the transformation (4.8) preserves the positive and negative subalgebras. The right differential of the last equation produces a solution of the dKP hierarchy (with solution $\tilde{u}(\tilde{x}, \tilde{\mathbf{t}})$ for the dKP equation) which is given by

$$\tilde{\omega}_+ = \mathrm{Ad}\tilde{\eta}_-\, \mathrm{d}\tilde{t}(\tilde{p})|_+ = \mathrm{d}\tilde{t}(\tilde{p}) + \{\ln\tilde{\eta}_-,\ \mathrm{d}\tilde{t}(\tilde{p})\}_+ + \cdots. \tag{4.9}$$

The sought transformation follows by the elimination of η_- between formulae (4.5) and (4.9) which for the solutions $u(x, \mathbf{t})$ and $\tilde{u}(\tilde{x}, \tilde{\mathbf{t}})$ becomes the explicit relation

$$\tilde{u}(\tilde{x},\tilde{\mathbf{t}}) = \frac{\partial\tilde{p}}{\partial p}\frac{\partial x}{\partial\tilde{x}}[u(x,\mathbf{t}) - u_0(x,\mathbf{t})] \tag{4.10}$$

where $u_0(x, \mathbf{t})$ denotes the solution of the dKP equation determined by the zero-curvature form $\mathrm{d}\xi_+\xi_+^{-1}$ already studied in the preceding section.

Proposition 4.1. *Let k_1 define the factorization problem (4.1) and assume that $\tilde{u}$ and u are the solutions to the dKP equation associated with the factorization problems (2.4) defined by the elements k and k_1k respectively. Then, $\tilde{u}$ is obtained from u according to formula (4.10).*

To write concrete examples we first set $r = 2$ and keep the notation conventions of section 3, $z_a = J_a(t)$ for $a = 1, 2, 3$. Then, from equations (3.10) and (4.8), we obtain the transformation for the canonical variables,

$$\tilde{p} = \frac{p}{(z_3')^{1/3}} - \frac{2yz_3''}{9(z_3')^{4/3}} - \frac{z_2'}{3z_3'} \qquad \tilde{x} = x(z_3')^{1/3} + \frac{2y^2z_3''}{9(z_3')^{2/3}} + \frac{2yz_2'}{3(z_3')^{1/3}} + z_1. \tag{4.11}$$

In addition, the time variables transform according to

$$\tilde{y} = y(z_3')^{2/3} + z_2 \qquad \tilde{t} = z_3$$

and the new solution to the dKP equation, as given by (4.10), is $\tilde{u}(\tilde{x}, \tilde{y}, \tilde{t}) = (z_3')^{-2/3}(u(x, y, t) + 2\phi_{1,x})$, with the function $\phi_{1,x}$ defined in formula (3.13). These are the transformations of [21, 12] up to a permutation of tilde and normal variables. One of the main achievements of the group theory for the dKP equation consists in the explanation of these rather involved transformation formulae. Analogous computations for $r = 3$, with $z_a = J_a(t_4)$ for $a = 1, 2, 3, 4$, lead to the following relations,

$$\tilde{p} = \frac{p}{(z_4')^{1/4}} - \frac{3tz_4''}{16(z_4')^{5/4}} - \frac{z_3'}{4z_4'}$$

and

$$\tilde{x} = (z_4')^{1/4}x + \frac{9t^3[4z_4'(z_4)^{(3)} - 3(z_4'')^2]}{512(z_4')^{7/4}} + \frac{9t^2[2z_4'z_3'' - z_3'z_4'']}{64(z_4')^{3/2}}$$
$$+ \frac{3t}{32(z_4')^{5/4}}[-(z_3')^2 + 8z_2'z_4' + 4y(z_4')^{1/2}z_4''] + \frac{yz_3'}{2(z_4')^{1/2}} + z_1$$

for the pair of canonical coordinates. The new time variables now take the form

$$\tilde{y} = (z_4')^{1/2}y + z_2 + \frac{1}{32(z_4')^{1/2}}[24tz_3'(z_4')^{1/4} + 9t^2z_4'']$$

and

$$\tilde{t} = (z_4')^{3/4}t + z_3 \qquad \tilde{t}_4 = z_4.$$

Finally, the resulting solution for the dKP equation is

$$\tilde{u} = (z_4')^{-1/2}u + \frac{1}{256(z_4')^{5/2}}[3t^2[15(z_4'')^2 - 12z_4'(z_4)^{(3)}] + 96t[(z_4')^{5/4}z_3'' - z_3'(z_4')^{1/4}z_4'']$$
$$- 64yz_4'z_4'' + 48(z_3')^2(z_4')^{1/2} - 128z_2'(z_4')^{3/2}].$$

5. Composition of transformations

The canonical transformation defined by the generating function (3.1) represents a particular case of more general transformations studied in [12]. Let us consider, for instance, the transformation defined by the generating function

$$J(P,\rho) = J_{r+1}(\rho)P^{\frac{r+1}{m}}$$

in the notation of section 3, with a fixed integer $m \geqslant 1$. Thus we have a map $(p,x) \stackrel{K}{\to} (P,X)$ characterized by the equations

$$p^{r+1} = J'_{r+1}(\rho)P^{\frac{r+1}{m}} \qquad X = \frac{r+1}{m}J_{r+1}(\rho)P^{\frac{r+1-m}{m}}.$$

We observe that this transformation decomposes into simpler constituents. For if we take $m = 1$ in the previous formulae, as in (3.1), we shall have $(p,x) \stackrel{K_1}{\to} (P_1, X_1)$ defined by

$$p^{r+1} = J'_{r+1}(\rho)P_1^{r+1} \qquad X_1 = (r+1)J_{r+1}(\rho)P_1^r.$$

If, in addition, we denote by $(p,x) \stackrel{K_m}{\to} (P_m, X_m)$ the transformation induced by the particular choice $J_{r+1}(\rho) = \rho = x/(r+1)p^r$, keeping fixed the value of m, we readily get

$$p^{r+1} = P_m^{\frac{r+1}{m}} \qquad X_m = \frac{x}{mp^r}P_m^{\frac{r+1-m}{m}}$$

which represents the well-known transformation [11]

$$P_m = p^m \qquad X_m = \frac{x}{mp^{m-1}}.$$

We conclude that the composition of the transformations K_1 and K_m results in $K = K_mK_1$, which amounts to the relations $P(p,x) = P_m(P_1,X_1)$ and $X(p,x) = X_m(P_1,X_1)$.

A generalization of the situation just described appears in connection with the generating function [12]

$$J(P,\rho) = \sum_{k=1}^{r+1} J_k(\rho)P^{\frac{k}{m}} + \sum_{k=r+2}^{m+n} \gamma_k P^{\frac{k}{m}} \tag{5.1}$$

where n is a non-negative integer. The associated transformation, we shall denote again by K, takes the canonical pair (p, x) into the new variables (P, X), $(p, x) \stackrel{K}{\to} (P, X)$, defined by the implicit equations

$$p^{r+1} = \sum_{k=1}^{r+1} J'_k(\rho) P^{\frac{k}{m}} \qquad X = \sum_{k=1}^{r+1} \frac{k}{m} J_k(\rho) P^{\frac{k-m}{m}} + \sum_{k=r+2}^{m+n} \frac{k}{m} \gamma_k P^{\frac{k-m}{m}}. \tag{5.2}$$

The transformation K_1, obtained from (5.2) when we set $m = 1$ and all of the $\gamma_k = 0$, is now $(p, x) \stackrel{K_1}{\to} (P_1, X_1)$, with defining equations

$$p^{r+1} = \sum_{k=1}^{r+1} J'_k(\rho) P_1^k \qquad X_1 = \sum_{k=1}^{r+1} k J_k(\rho) P_1^{k-1} \tag{5.3}$$

as in section 3. The definition of K_m becomes in the present situation, $(p, x) \stackrel{K_m}{\to} (P_m, X_m)$,

$$p^{r+1} = P_m^{\frac{r+1}{m}} \qquad X_m = \frac{x}{mp^r} P_m^{\frac{r+1-m}{m}} + \sum_{k=r+2}^{m+n} \frac{k}{m} \gamma_k P^{\frac{k-m}{m}} \tag{5.4}$$

which admits the explicit solution given by the formulae

$$P_m = p^m \qquad X_m = \frac{x}{mp^{m-1}} + \sum_{k=r+2}^{m+n} \frac{k}{m} \gamma_k p^{k-m}. \tag{5.5}$$

Proposition 5.1. *The transformation (5.2) is the result of the composition of (5.5) with (5.3), that is* $K = K_m K_1$.

Proof 5.1. The composition gives $P_m = P_1^m$ which implies the first of equations (5.2) as a consequence of the defining relation for P_1 in (5.3). Analogously, for X_m we have

$$X_m = \frac{X_1}{m P_1^{m-1}} + \sum_{k=r+2}^{m+n} \frac{k}{m} \gamma_k P_1^{k-m}$$

which in terms of P_m reads

$$X_m = \sum_{k=1}^{r+1} \frac{k}{m} J_k(\rho) P_m^{\frac{k-m}{m}} + \sum_{k=r+2}^{m+n} \frac{k}{m} \gamma_k P_m^{\frac{k-m}{m}}.$$

Therefore, the pair (P_m, X_m) is a solution for the system (5.2). □

Conjugation with constant elements in the group preserves the solutions to the factorization problem. According to proposition (4.1), left and right composition with K_1, which are induced by left and right translations by k_1 respectively, are then symmetries for the dKP hierarchy. Therefore, solutions of the dKP hierarchy obtained from the twistor equations through the canonical transformation (5.2) reduce, by proposition (5.1), to solutions determined by formulae (5.5). These are a particular instance of transformations of the following type,

$$P = P(p) \qquad X = \frac{x}{P'(p)} + \gamma(p) \tag{5.6}$$

where $P(p)$ and $\gamma(p)$ are arbitrary functions which do not depend on x. The solutions furnished by the twistor equations for this type of transformations are easily described upon further specialization of the function $P(p)$. The class of transformations (5.6) defined by functions of the form

$$P(p) = p^m \, \mathrm{e}^{A_-(p)} \tag{5.7}$$

for an integer $m \geqslant 2$ and a function $A_-(p)$ strictly negative in powers of the variable p, gives rise to a corresponding collection of dKP solutions characterized by a finite number of conditions. In the present context we shall not assume any restriction about the number of time variables figuring in the formulae under consideration. Because $P(p)$ does not depend on x, the first of the twistor equations (2.7) is identically satisfied provided $L(p, x)$ is implicitly defined through the relation

$$L^m \, \mathrm{e}^{A_-(L)} = p^m + w_1 p^{m-1} + \cdots + w_{m-1} p + w_m \tag{5.8}$$

where the coefficients w_k are functions of x and $\mathbf{t}$ with w_1 being constant. This fixes Φ_x and must be completed with the compatibility conditions that follow from the second of the twistor equations (2.7). According to proposition 2.1 this amounts to the vanishing of the first $m-1$ coefficients of $G_-(p, x)$ and for a prescribed generating function $\Phi(L, x)$ as in (2.6) results in the set of equations

$$\int \left[\frac{t'(L) + x}{P'(L)} + \gamma(L) \right] p^{k-1} \, \mathrm{d}p = 0 \qquad k = 1, 2, \ldots, m-1 \tag{5.9}$$

for a closed path around the origin. In these equations $L(p, x)$ is given in terms of p as defined by equation (5.8). Solving (5.8) for p we can express relations (5.9) in the form

$$\int \left[\frac{t''(L)P'(L) - (t'(L) + x)P''(L)}{P'(L)^2} + \gamma'(L) \right] p(L, x)^k \, \mathrm{d}L = 0 \tag{5.10}$$

for $k = 1, 2, \ldots, m-1$. With this representation the influence of (5.8) reduces to the substitution in the integral of the first $m-1$ powers of $p(L, x)$ instead of the longer computation needed for (5.9). Greater simplification is gained with the restriction in the number of the time variables. It is not hard to see that if $t(L)$ is a polynomial in L instead of a series with an infinite number of positive powers of L, and in the function $\gamma(p)$ the number of positive powers of p is finite, relations (5.10) define the solution by a system of algebraic equations. An example of this situation is the transformation

$$P(p) = p^4 \left(1 + \frac{v}{5p^5} \right) \qquad X(p, x) = \frac{x}{P'(p)}$$

where v is a parameter; we set the time variables by means of the function

$$t(p) = yp^2 + tp^3 + \frac{1}{10} s p^{10}$$

in order to appreciate the influence of higher times in the solution of the dKP equation. System (5.10) consists in three equations that can be reduced to the pair of conditions

$$\frac{256s}{27} v^4 + (64x + 96tu - 6\nu s u^2)v + \frac{s\nu^2}{2} u - 12\nu t = 0$$

$$-\frac{256s^2}{9} u v^4 + \frac{32\nu s^2}{9} v^3 + \frac{256sy}{3} v^2 + \frac{\nu^2 s^2}{4} u^2 - 12\nu s t u + 144t^2 = 0$$

with $u = -2\phi_{1,x}$ and $v = 3\phi_{2,x}$, which furnish the desired solutions u of the dKP equation.

Acknowledgment

FG was partially supported by the CICYT proyecto no BFM2002-01607.

References

[1] Lebedev D and Manin Yu 1979 *Phys. Lett.* A **74** 154
[2] Zakharov V E 1980 *Func. Anal. Priloz.* **14** 89
Zakharov V E 1981 *Physica* D **3** 193
[3] Zakharov V E 1994 Dispersionless limit of integrable systems in 2 + 1 dimensions *Singular Limits of Dispersive Waves (Nato Adv. Sci. Inst. Ser. B Phys.* vol 320*)* ed N M Ercolani *et al* (New York: Plenum)
[4] Krichever I M 1992 *Commun. Pure Appl. Math.* **47** 437
[5] Krichever I M 1989 *Funct. Anal. Appl.* **22** 200
Krichever I M 1989 *Commun. Math. Phys.* **143** 415
[6] Kupershmidt B A 1990 *J. Phys. A: Math. Gen.* **23** 871
[7] Saveliev M V 1992 *Theor. Math. Phys.* **92** 457
[8] Gibbons J and Tsarev S P 1996 *Phys. Lett.* A **211** 19
Gibbons J and Tsarev S P 1996 *Phys. Lett.* A **258** 263
[9] Mineev-Weinstein M, Wiegmann P B and Zabrodin A 2000 *Phys. Rev. Lett.* **84** 5106
[10] Wiegmann P B and Zabrodin A 2000 *Commun. Math. Phys.* **213** 523
[11] Takasaki T and Takebe T 1992 *Int. J. Mod. Phys.* A **7** 889
Takasaki T and Takebe T 1995 *Rev. Math. Phys.* **7** 743
[12] Guil F, Mañas M and Martinez Alonso L 2003 *J. Phys. A: Math. Gen.* **36** 6457
[13] Kodama Y 1988 *Phys. Lett.* A **129** 223
Kodama Y 1988 *Prog. Theor. Phys. Suppl.* **95** 184
[14] Kodama Y and Gibbons J 1989 *Phys. Lett.* A **135** 167
[15] Geogdzhaev V V 1985 *Sov. Phys.—Dokl.* **30** 840
Geogdzhaev V V 1987 *Physica* D **87** 168
[16] Kodama Y 1990 *Phys. Lett.* A **147** 477
[17] Konopelchenko B and Martinez Alonso L 2001 *Phys. Lett.* A **286** 161
[18] Konopelchenko B, Martinez Alonso L and Ragnisco O 2001 *J. Phys. A: Math. Gen.* **34** 10209
[19] Mañas M, Martinez Alonso L and Medina E 2002 *J. Phys. A: Math. Gen.* **35** 401
[20] Guil F, Mañas M and Martinez Alonso L 2003 *J. Phys. A: Math. Gen.* **36** 4047
[21] Dunajski M, Mason L J and Tod P 2001 *J. Geom. Phys.* **37** 63
[22] Dunajski M and Tod P 2002 *Phys. Lett.* A **303** 253